Gibrat's Law and Diversification*

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Abstract

This paper presents an analysis of the growth patterns of the worldwide top 150 firms in the pharmaceutical industry. A test of the Gibrat's Law of Proportionate Effect is performed and we find, in line with previous literature, a violation concerning the variance of growth. Using disaggregated data on submarkets we are however able to show that this violation can be completely accounted for by a diversification effect, namely a scale relation between the number of sub-markets in which a firm is active and its size. To interpret these findings we propose a stochastic branching model of firms diversification consistent with a notion of cumulative corporate competences.

JEL Codes: L1,C1,D2

Keywords: branching process, diversification, Gibrat Law, cumulative corporate competences

1 Introduction

This work reassesses the "stylized fact" concerning the relationship between variance in corporate growth rates and corporate size employing empirical evidence from the world pharmaceuticals industry. A statistical analysis using novel disaggregate data is shown, which suggests an interpretation grounded on corporate diversification patterns characterized by stochastic accumulation of competencies across an increasing number of product markets.

The point of departure of the present analysis is the "random walk" description of firm growth (c.f. Gibrat (1931) and, within an enormous literature, Simon and Bonini (1958); Hart (1962); Ijiri and Simon (1977); Sutton (1997); Brock (1999); Geroski (2000)). The baseline stochastic model of firms growth proposed by Gibrat is known as the "Law of Proportionate Effect" (LPE), and relates the size of a firm and its rate of growth by the expression

$$S(t+1) = S(t)\,(1 + R(t)) \tag{1}$$

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where $S(t)$ is the firm's size at time t and R is a random variable not dependent on S. This process reduces to a random walk in the log of size $s(t) = \log(S(t))$ and (under the usual assumption of validity of the Central Limit Theorem) predicts an asymptotic log-normal size distribution. Even if the "crude" version of the model misses many important aspects of industrial dynamics (including firms entry, exit, merging, etc.) it provides a robust benchmark framework for the interpretation of data.

The evidence, as explored by a rich and growing literature (see, for instance Mansfield (1962); Evans (1987); Hall (1987) and more recently Stanley et al. (1997); Lee et al. (1998)) roughly yields two, apparently conflicting, conclusions. On one hand, the Gibrat hypothesis is confirmed by the lack of any relationship between the (log) size of firms and their average rate of growth. On the other hand, it is violated by a clear negative dependence of the growth variance on size.

It has been suggested that a natural explanation for the reduction of growth variance with size could be a sort of "portfolio" effect (see eg. Hymer and Pashigian (1962)). The basic idea is that a firm can be described as a collection of "atomic" components (line of productions, plants, etc.) of roughly the same size whose number would be proportional to the size of the firm. Under the assumption that the growth processes for different components are independent, the Law of Large Number (LLN) would predict a decrease in the variance of the firm growth rates proportional to the inverse squared root of its size. The observed dependence is however milder and the failure of the LLN is usually inputed (see e.g. Boeri (1989)) to the existence of a "relation" between the firm components that makes the aggregation of the "atomic" growths not simply additive[1]. A different model has recently been proposed (Stanley et al., 1997; Lee et al., 1998), based on a supposed intra-firm complex hierarchical structure, which, opportunely tuned, reproduce quite well the observed behavior. On the contrary, it will be shown in what follows that, if one correctly identifies the "atomic" contributions to growth and the scaling relation between their number and the firm size, the LLN does a good job in explaining the observed Gibrat violation without any need of intra-firm "structure"[2], at least as far as the database under analysis is concerned.

The analysis presented in this paper concerns the top 150 firms operating in the seven major western markets (USA, United Kingdom, France, Germany, Spain, Italy and Canada) in the period ranging from 1987 to 1997.

The remainder of the paper is organized as follows. In Section 2 we present a brief overview of the database used and of the statistical properties of both the firms size distribution and growth process. For more details one can refers to Bottazzi et al. (2000, 2001), where a wider statistical description of the data is presented. On the contrary, here we concentrate the attention on the sole aspect of the violation of the Gibrat model emerging as a relation between the variance of growth and the size of a firm. Section 3 performs an analysis of this relation, using disaggregated data on the "sub-markets" defined by the 4-digit ATC code. We are then able to reduce the Gibrat violation to a "diversification" effect, due to a scale relationship between the firm size and the number of its active markets. Finally in Section 4, we propose a model, inspired by the structure of the "competence accumulation" dynamics in the pharmaceutical industry, which accounts for the observed pattern of firm diversification.

[1] The sole introduction of correlation in growth components is clearly not enough.

[2] After the first draft of this work had been completed the author has become aware of an analogous conclusion drawn in Amaral et al. (1998). The model presented there, even if similar in spirit to the present one, requires more demanding assumptions on the underlying structure of the corporate growth process and relies, in order to provide estimates comparable with available data, on extensive numerical simulations.

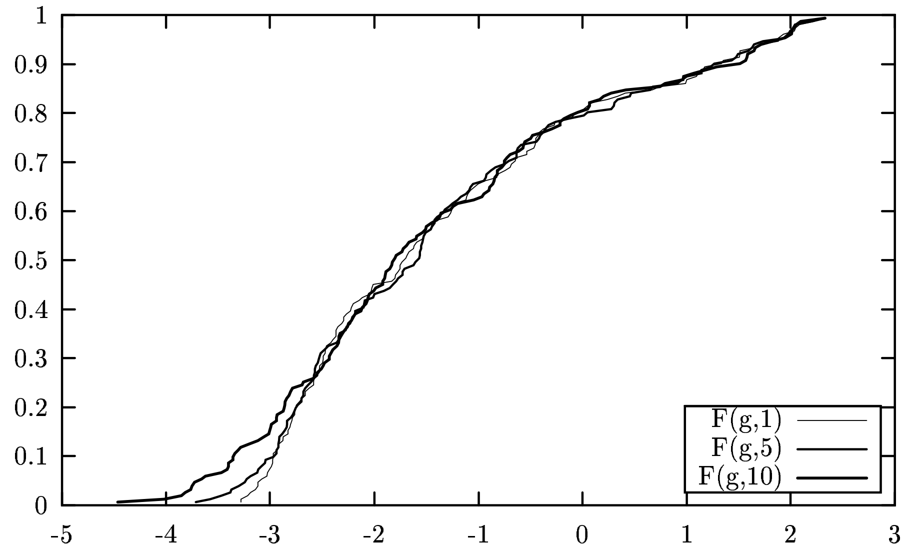

Figure 1: Distribution function for the firm "normalized size" g. The three curves correspond to the distribution $F(g,t)$ observed at three different years: 1987($t=1$), 1992 ($t=5$) and 1997 ($t=10$).

2 The database

The present statistical analysis is based on the dataset PHID, Pharmaceutical Industry Database[3] that covers sales figures for the top incumbents in the seven major western markets (USA, United Kingdom, France, Germany, Spain, Italy, Canada) for a time frame of 10 years. Aggregate sales by each firm stand for the sum of its sales in each of the national markets. We consider the top 150 firms obtained from the intersection of the top 100 firms (in terms of sales) in each national market, at the beginning of the period of observation. The choice of the time-zero ranking is obviously meant to avoid a sample selection bias in favor of the most successful ones, even if at the cost of censoring some (in actual fact very few) entrants among the top ranks. The analysis is constrained to only the top 150 worlwide firms for several reasons of statistical cleanliness, the most important being the possibility that the lower ranks include national firms which are "big" in one single market but smaller than other firms, left out of the sample, operating on other markets.

The database contains information about sales of 7654 drugs by each firm from 1987 to 1997. These single product sales are aggregated in 517 microclasses according to the 4-digit-level of the Anatomical Therapeutic Classification scheme (ATC) introduced by the World Health Organization. This scheme classifies drugs according to the anatomical part of the human body they affect together with the targeted deseases. Since the substitutability of products belonging to different classes is practically zero, this classification provides an almost perfect definition for the different markets.

This work is focused on the processes of *internal* growth and diversification. Hence, to take into account mergers and acquisitions during the period of observation, we have constructed "super firms" which correspond to the end-of-period actual entity (so for example, if any two

[3]This database was originally developed at the University of Siena and is now mantained and improoved at the University of Florence by Fabio Pammolli, Massimo Riccaboni and collaborators.

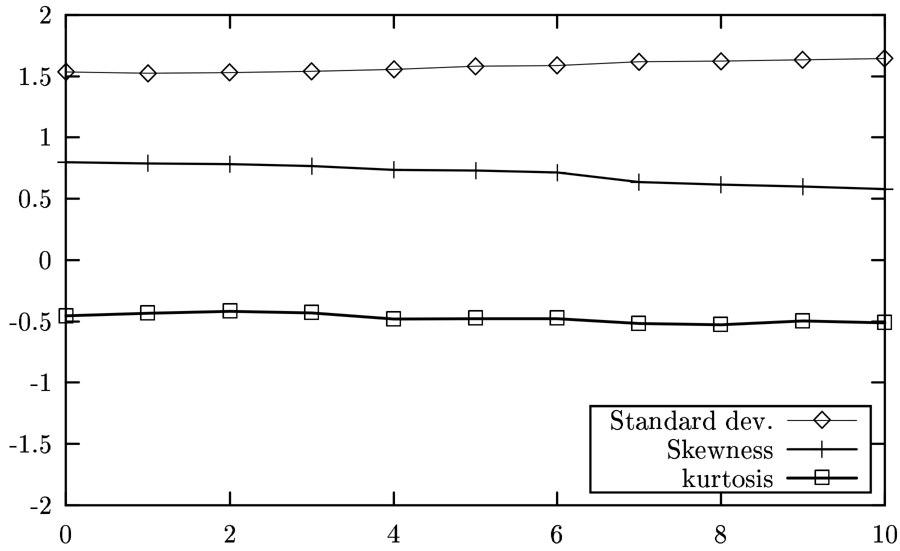

Figure 2: Moments of the g distribution computed at different times

firms merged during the observed history, we consider them merged from the start). This procedure clearly biases intertemporal comparisons on actual size distributions, but it helps to highlight those changes in the distributions which are due to processes of intra-market competition and inter-market diversification.

Given the overwhelming problems of deflation in this industry, all sales in the database are in USD at current prices and exchange rates: therefore, the analyses that follow will all entail some normalization procedure (e.g. in terms of market shares or equivalent measures) such as to get rid of the effect of price changes.

Let $S_i(t)$ be the size of firm i ($i \in [1, \ldots, N]$) at time t ($i \in [0, \ldots, T]$) where (see Sec. 1) $N = 150$ and $T = 10$ and define the 'normalized sizes" as $G_i(t) = S_i(t) / < S_i(t) >$. This variable and its logarithm $g_i(t) = \log(G_i(t))$ are characterized by distribution functions that can be considered stationary in time (see Fig. 1). The reliability of this assumption may be checked by plotting the moments of the distribution as a function of time (see Fig. 2): apart from a mild increase in the variance of the distribution, the approximation appears quite good[4].

As a first step in the search for a violation of the LPE it is natural to investigate the possible presence of a relation between the first moments of the distribution of growth shocks and size. Having defined $h_i(t) = g_i(t+1) - g_i(t)$, the growth of firm i at time t, one bins all the firms according to their size $g_i(t)$ in equally populated sets, and computes the mean, standard deviation and 1-lag autocorrelation of the growth $h_i(t)$ separately for each set. The results are reported in Fig. 3 against the average size of each bin.

Both the mean growth and the autocorrelation do not show any particular pattern but a clear dependence of the growth variance on size emerges revealing a "second order" violation of The Law of Proportionate Effect. Fitting the relation between variance of growth and size

[4]Notice that the major contribution to the variance increase comes clearly from the low g region (see Fig. 1) where a "spurious" (due to the selection bias) diffusion toward smaller sizes is present. This diffusion is responsible for the observed increase in variance. Once understood, this effect is statistically irrelevant and in the following analysis the g variable is thought identically distributed at every time step.

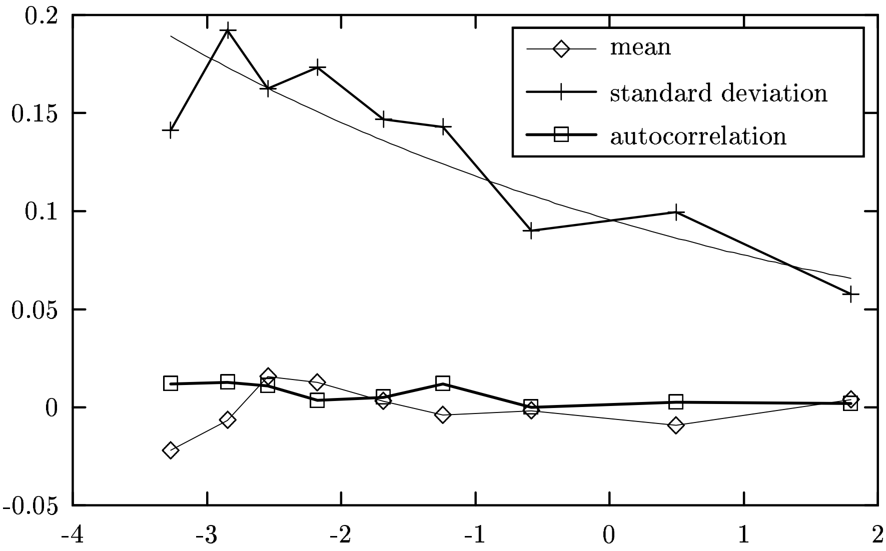

Figure 3: Mean, standard deviation and autocorrelation of growth h computed for different size bins plotted against the average size in the bin. The exponential fit to the standard deviation gives a value $\gamma = -0.20 \pm 0.03$

with an exponential law

$$\sigma(h) \sim e^{-\beta g} \tag{2}$$

one finds a value $\beta = 0.2 \pm .03$ that is strikingly similar to the one found in other analysis on different datasets (Stanley et al., 1997; Lee et al., 1998) and much lower than $\beta = 0.5$ which would be predicted on the ground of a theory of firm size based on a sheer "portfolio" aggregation of independent "atomic" components. However, the fundamental point that we shall argue in the following is that the LLN is indeed sufficient to completely characterize the relation (2) whenever the growth of the firm as a whole is interpreted as an "aggregation" of its growth in the different sub-markets in which it operates.

3 Diversification as a source of Gibrat violation

One of the peculiar features of the PHID database is, as mentioned, the possibility of disaggregating the sales figures until the 4th digit of the ATC code. This level of disaggregation allows to identify sub-markets that are "specific" enough to be considered, roughly speaking, the *loci* of competition between firms: the products belonging to a given sub-market possess similar therapeutic characteristics and can then be considered substitutable while products belonging to different sub-markets are usually targeted to different pathologies. Moreover, Orsenigo et al. (2000) suggest that this disaggregation level is also the one at which single research projects develop. These phenomena support the hypothesis that the different therapeutic sub-markets provide the natural "scale" at which firm growth should be studied.

Given that, here we shall consider in detail the variance-size relation characterizing the process of growth. In order to disentangle such a relation, one needs, first, to study the correlation across submarkets, and second, to analyze the relation between the number of submarkets in which a firm operates and its size. For this purpose let us introduce some notation.

Let $S_{i,j}(t)$ be the size of firm i in sub-market j at time t. The aggregate size of the i-th firm is the sum of its size on all the sub-markets $S_i(t) = \sum_j S_{i,j}(t)$ and defining the rescaled variables $G_{i,j}(t) = S_{i,j}(t) / < S_i(t) >$ the aggregate growth $H_i(t) = G_i(t+1)/G_i(t)$ can be rewritten as

$$H_i(t) = \sum_j \frac{G_{i,j}(t+1)}{G_i(t)} = H_{i,j}(t)\ \Delta_{i,j}(t)\ \frac{1}{N_i(t)} \tag{3}$$

where the contribution of each sub-market has been factorized in three terms: $H_{i,j}(t) = G_{i,j}(t+1)/G_{i,j}(t)$, which is the actual growth of the firm i in sub-market j; the inverse number of active markets $1/N_i(t)$ and $\Delta_{i,j}(t) = N_i(t)\, S_{i,j}(t)/S_i(t)$, a weighting coefficient capturing the "diversification heterogeneity" of a firm. If the firm i at time t is symmetrically diversified over its sub-markets, the values of $\Delta_{i,j}(t)$ for different j are concentrated around 1, otherwise if the firm is asymmetrically diversified, the $\Delta_{i,j}(t)$ are broadly distributed.

We are interested in the study of the aggregate growth rates $H_i(t)$ and having rewritten them in terms of sub-market contributions $H_{i,j}(t)$ seems at a first glance to have complicated the matter, sice we have now to deal with the sum of a large number of variables. On the other hand, considering the cross-correlations

$$C_i(j,j') = \left\langle \frac{G_{(i,j)}(t+1)}{G_i(t)} \frac{G_{(i,j')}(t+1)}{G_i(t)} \right\rangle_t \tag{4}$$

for all the firms in all the sub-markets, i.e. among all the possible couples of indeces (i,j) and (i,j'), one obtains a distribution sharply centered around zero, with a standard deviation of 0.000388 and an average deviation of 0.000024. As a plausible approximation we can then consider $C_i(j,j') \sim 0$ so that the growth processes on different sub-markets can be considered to any extent uncorrelated. In particular the variance of the aggregate (firm) growth is obtained adding the variance of the growth in each sub-market and one is allowed to write

$$\mathrm{var}_{i,t}[H_i(t)] = \sum_j \mathrm{var}_{i,t}\left[H_{i,j}(t)\ \Delta_{i,j}(t)\ \frac{1}{N_i(t)}\right] \tag{5}$$

where $\mathrm{var}_{i,t}$ denotes the variance of the distribution computed , consistently with the previous analysis, using the complete panel (all the firms at all the time steps).

From (5) we see that the possible relationship between the variance of aggregate growth rates and firms size is likely to come from different directions. Indeed, in the left hand size appears the product of three different variables that can all be plausibly related to the firm size. For instance, a dependance of $H_{i,j}$ on size could came from the existence of a sort of "increasing returns to aggregate scale" while an analogous dependance for $\Delta_{i,j}$ could be originated by a different business structure implied by a different degree of "corporate coherence" (in the spirit of Teece et al. (1994)) for firms with different market power. These effects, however, seem to lack in our data. Indeed, if one considers the mean and variance of the variables $H_{i,j}(t)$ and $\Delta_{i,j}(t)$ for different size bins, no clear dependence on the average size of the bin appears, as can be seen in Fig. 4.

Therefore, the term $N_i(t)$, the number of active sub-market a firm possesses, is presumibly the only responsible for the observed dependence of variance over the aggregate size. Fitting on a log-log scale the average number $< N_i(t) >$ of active markets for each bin against the average size of the bin (see Fig. 5) one obtains a slope $\alpha = .39 \pm .02$ and an intercept $q = 6 + -0.12$. The Law of Large Numbers would predict a relation between the exponent in (2) and the slope

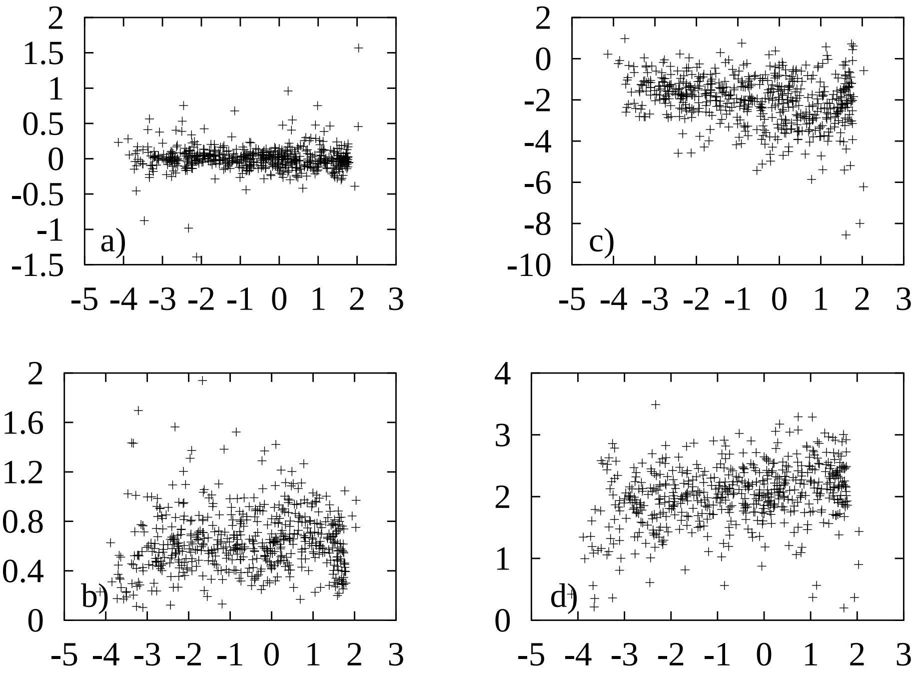

Figure 4: Results of various statistics computed on groups of firms binned according to their size. On the x-axis is the log of the average size of the bin. **(a)** Mean and **(b)** variance of $H_{i,j}(t)$ for different sub-markets j. **(c)** Mean and **(d)** variance of $\Delta_{i,j}(t)$ for different sub-markets j.

in Fig. 5 of the form $\beta = -\alpha/2$ which is in perfect agreement with the data[5].

The conclusion is that the explanation of the relationship between the variance of growth rates distribution and the size of firm based on the Law of Large Number is valid, as long as one consider the actual number of sub-markets a firm operates in. It must be stressed, however, that in demostrating this statement, we also ruled out two other possible sources of functional dependence between a firm's size and the variance of its aggregate growth: first, the possibility that the variance or the mean of a firm growth in a given sub-market depends (on average) on its total size and, second, the possibility that the diversification pattern of a firm (described by the variable Δ) could vary (on average) with its size. Both these possibilities are actually discussed in the literature: cf., for instance, Hart and Prais (1956) who propose them as possible sources of Gibrat Law violation.

It remains however to explain why the number of active markets a firm possesses shows such a clear dependence on its size $N(G) \sim G^{\lambda}$ and what is the meaning (if any) of the parameter λ. This is what we shall do in the next Section where we propose a model of the diversification process, essentially based on the assumption of a self-reinforcing effect cumulatively driving the "proliferation" of sub-markets a firm operates in, which provide a simple but suggestive interpretation of Fig. 5.

[5]Notice that a weak relation appears between the variance of $\Delta_{i,j}$ and the total size. A linear fit provides a slope .01 that is negligible if compared to the effect due to the number of active sub-markets.

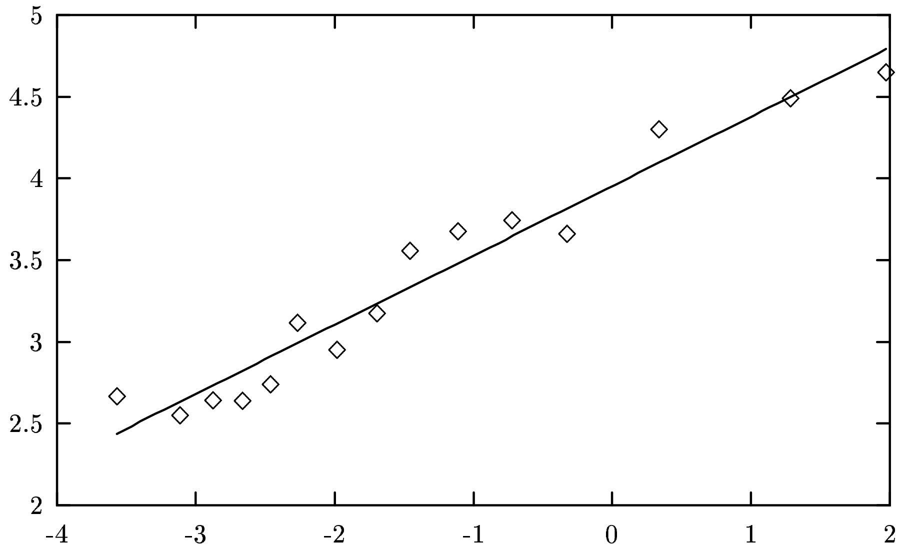

Figure 5: Average number of active sub-markets. The fit is a straight line with slope $\alpha = .41 + -.02$.

4 A stochastic branching model for firm diversification

The previous analysis shows that larger firms are present, on average, on a greater number of sub-markets. For the purposes of this Section it is useful to read this statement in a dynamical fashion: as firms grow their activities become more and more diversified since they sell products on different sub-markets.

In what follows we shall try to capture the diversification behavior of firms with a model describing how the number of active markets, or better, how the probability of having a given number of active markets changes as the firm grows.

Given the nature and limitations of our data, we consider the firm size as the independent variable and analyze the changes of diversification patterns in accordance to changes in the former[6].

The object of the following analysis thus becomes the probability that a firm which possesses n active markets when its (log) size is g, will possess m active markets when its (log) size is $g' \geq g$, $P(n, g; m, g')$[7].

Before proceeding to build the model, let us briefly mention a few contributions which have attempted to merge the diversification and the growth aspect of firm's dynamics. This class of models, collectively referred as "island models", originally proposed by Simon (see Ijiri and Simon (1977) and the reassessment in Sutton (1998)), does not explicitly address the diversification dynamics but rather considers that part of a growth process driven by a successive capture of diverse "islands", or "business opportunities". In this way the distinction between

[6]Note that this approach is distinct from considering a "dynamical" process in time, i.e. a model describing the evolution of diversification patterns "as time goes by". A discrete-time model of such kind has been proposed and numerically analyzed in Amaral et al. (1998). Even if this kind of model presents many interesting features we prefer to study a "cross sectional" model since the former turns out to be sensitive to finer specifications of growth dynamics which can be hardly checked against available data.

[7]Due to the multiplicative nature of the growth, the natural candidate to play the role usually assigned to "time" in this kind of random processes, is the log of firms size g.

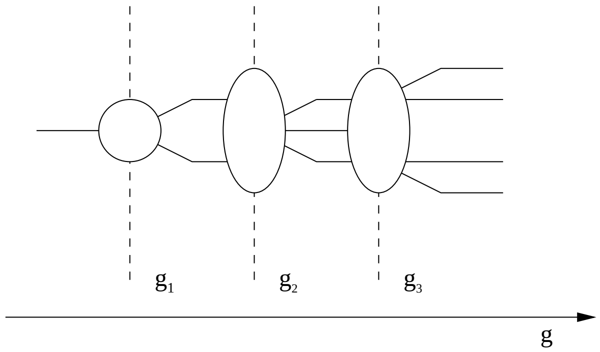

Figure 6: Diversification by independent events. The interarrival "growths" $g_i - g_{i-1}$ are exponentially distributed.

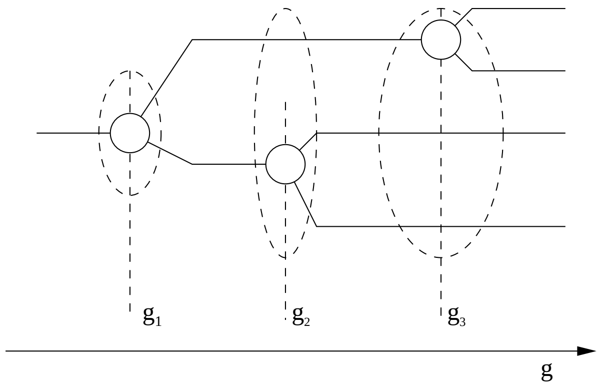

Figure 7: Diversification by branching. The interarrival "growths" are no longer exponential, the distance between two events decreases as the number of active markets increases.

"growth" and "diversification" becomes blurred, but nevertheless two assumptions are generally made concerning the latter. The first one concerns the nature of the "diversification" events (i.e. the entry on a previously unexploited market) and assumes that these events are seen as mutually independent "shocks" (investment opportunities, technological discoveries, etc.) that can "happen" to firms along their histories. The second assumption is that these "shocks" are uncorrelated in time, i.e. they could happen with the same constant probability at any moment, irrespectively to the actual firm "state". If one neglects the possibility of "instantaneous" multiple shocks[8], these assumptions provide a complete definition of the transition probability $P(n, g; m, g + \delta)$ between two sizes differing by an infinitesimal quantity δ:

$$P(n, g; m, g + \delta) = \begin{cases} \lambda\,\delta + o(\delta) & m = n + 1 \\ o(\delta) & m > n + 1 \end{cases} \quad . \tag{6}$$

where λ is the constant rate of arrival of "diversification events" (i.e. the average number of "diversification events" for unit time). This form of the transition probability defines the well-known Poisson process. A pictorial illustration of this process is shown in Fig. 6: the "blobs" stand for the diversification event "black boxes" that a firm meets along its growth. As far as this model is concerned, it is irrelevant what really happens inside the black box, what matters is that after the "shock" the firm ends up with one more active sub-market.

The Poisson process would however predict a linear increase of the average number of active sub-markets with g, a property that is clearly in contrast with the empirical results discussed in the previous Sections[9]. In order to obtain a more satisfying model, it proves to be useful to "open up the black boxes" of Fig. 6, trying to describe, at least in first approximation, the actual nature of the diversification events. A plausible interpretation of these events draws upon the extensive literature on the importance of technological capabilities and learning for corporate growth and diversification (for general arguments see Dosi (1988) and Teece et al. (1994) and in particular on the pharmaceutical industry see Orsenigo et al. (2000)).

[8]Equivalently: requires the property of orderliness for the associated stochastic process. For a general discussion on the complete characterization of point processes see Snyder (1975).

[9]Notice that one could also account for the observed behavior using a Poisson process with a non linear intensity function $\tau(G) = G^{\lambda}$ (see Snyder (1975)). This approach seems however rather *ad hoc* and would provide however a worse description for the overall statistics, see Fig. 8

Indeed, the dynamics of the pharmaceutical industry is largely research-driven, so that product innovations and imitations constitute essential aspects of the growth of firms and generate an intense spreading process across different markets(Bottazzi et al., 2000, 2001). Moreover, the development of technological capabilities appears to be an incremental and cumulative process, involving, in the case of pharmaceuticals, also "horizontal" innovative capabilities applicable across therapeutic targets (cf. also the indirect evidence stemming from patent agreement in Orsenigo et al. (2000)).

Diversification events are no exception: more often, a firm enters a new market when it has a technological capability of developing products for that market. It is then quite natural to expect that these patterns of technological accumulation will shape the growth dynamics of firms with the introduction of a "scope economy to diversification": the capability of a firm to enter a new market increases with its past successful diversification events[10].

A straightforward way of accounting for the presence of a "scope economy to diversification" is by supposing that the diversification process proceeds as a branching process, where each opened branch (sub-market) becomes eventually the origin of a new branching (diversification) event. A picture of this process is shown in Fig. 7 under the (minimal) assumption that the branching is uniformly binary. If one neglects the topology in sub-markets space that emerges from these successive branchings, but is interested only in the actual number of these sub-markets, the process in Fig. 7 can be readily described: all the active sub-markets can be sources of a possible "diversification" event *à la* Poisson, and (6) must be modified to read

$$P(n, g; m, g+\delta) = \begin{cases} n\,\lambda\,\delta + o(\delta) & m = n+1 \\ o(\delta) & m > n+1 \end{cases} \tag{7}$$

neglecting again multiple instantaneous branchings. If $p_n(g)$ is the probability that a firm of size g has n active markets it must satisfy the (pure-birth) set of equations

$$\begin{array}{rcll} p_n(g+\delta) & = & p_{n-1}(g)\,P(n-1, g; n, g+\delta) + p_n(g)\,P(n, g; n, g+\delta) & n > n_0 \\ p_{n_0}(g+\delta) & = & p_{n_0}(g)\,P(n_0, g; n_0, g+\delta) & n = n_0 \end{array} \tag{8}$$

where n_0 is the initial number of active markets. Substituting in (8) the definition given in (7) and taking the limit $\delta \to 0$ one obtains the set of differential equations:

$$\begin{array}{rcll} p'_n(g) & = & -n\,\lambda\,p_n(g) + (n-1)\,\lambda\,p_{n-1}(g) & n > n_0 \\ p'_{n_0}(g) & = & -n_0\,\lambda\,p_{n_0}(g) & n = n_0 \end{array} \tag{9}$$

with the initial conditions:

$$p_n(g_0) = \begin{cases} 1, & n = n_0 \\ 0, & n \neq n_0 \end{cases} \tag{10}$$

where g_0 is the initial size of the firm. This process is known as a Yule process and has been originally proposed to explain the proliferation in time of animal species (for a discussion see Feller (1968) and the references therein). The system (9) can be solved (see Appendix 1) to obtain the following distribution:

$$p_n(g) = \binom{n-1}{n-n_0} e^{-n_0\lambda(g-g_0)} \left(1 - e^{-\lambda(g-g_0)}\right)^{n-n_0} , \tag{11}$$

[10] Exceptions to the "technology driven" diversification events are purely commercial distribution agreements which, although not uncommon, are not dominant driver of diversification patters for the firms considered here. Nevertheless their dynamics too can plausibly be characterized by some sort of "scope economy" such that their presence does not in principle spoil the present analysis.

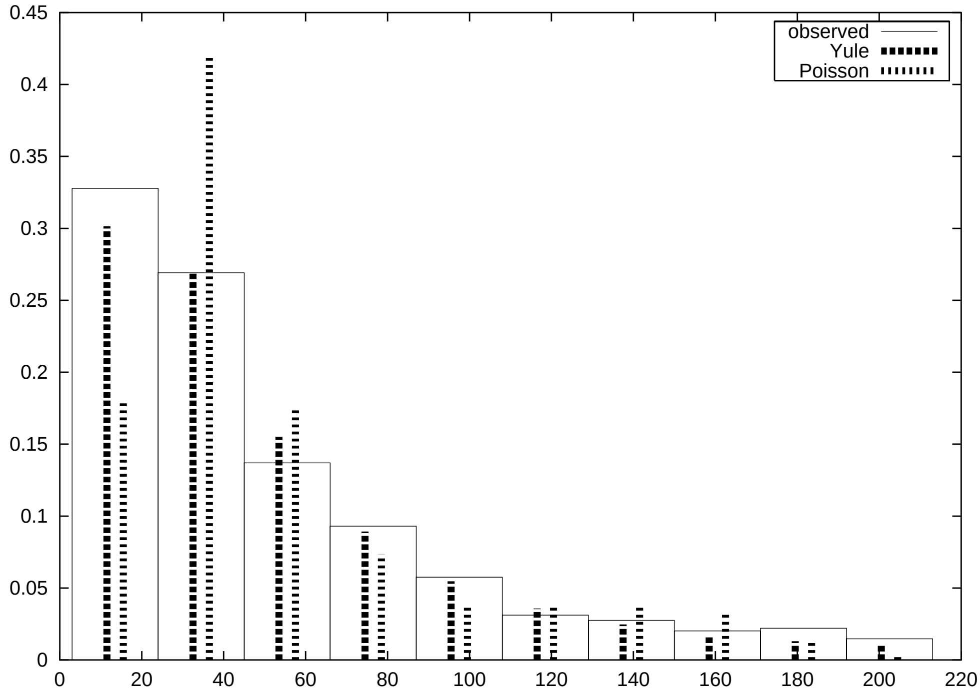

Figure 8: The binned probability density for the number of sub-market computed directly from the data and theoretically predicted by the Yule process. The theoretical distribution is characterized by $\lambda = .39$, $n_0 = 5$ and $s_0 = -12$. As comparison, a fit of a Poisson model with a non linear intensity function $\tau(G) = G^\lambda$ is also shown.

defined for $g \geq g_0$ and $n \geq n_0$.

Let us turn now to the problem of describing the observed data with the proposed model. The distribution in (11) contains three parameters: the "diversification" rate λ, the initial number of active sub-markets n_0 and the initial firm size s_0. Since it correctly predicts an exponential increase in the average number of active markets with size

$$< N(g) >= \sum_{n=n_0} n p_n(g) = n_0 e^{\lambda(g-g_0)} \ , \tag{12}$$

one can use the linear fit shown in Fig. 5 to obtain an estimate of the parameters in (12), requiring the fulfillment of the two conditions $\lambda = \alpha = .39$ and $\log(n_0) - \lambda s_0 = q = -6$.

The descriptive power of the model, however, must be judged using a more discriminating object than the average $< N(s) >$. A good candidate is the diversification pattern of the whole industry. To be precise, let $P(M_0 < M < M_1)$ be the probability that a firm possesses a number of active markets between M_0 and M_1. One way of computing this quantity is through the actual frequency of occurrences:

$$P^{ob}(M_0 < M < M_1) = \frac{1}{TN} \sum_{m=M_0}^{M_1} \sum_{i=1}^{N} \sum_{t=0}^{T-1} \delta(n_i(t) - m) \tag{13}$$

where $n_i(t)$ is the number of active markets of firm i at time t and δ is the Kronecker delta.

An alternative approach for the computation of the same object is via the convolution

$$P(M_0 < M < M_1) = \sum_{m=M_0}^{M_1} \int_{g_0}^{+\infty} dg f(g)\, p_m(g) \tag{14}$$

once the size probability density $f(s)$ and the "diversification" probability $p_n(g)$ are known. An estimate of (14) can be obtained using the observed frequencies in data

$$P^{th}(M_0 < M < M_1) = \frac{1}{TN} \sum_{m=M_0}^{M_1} \sum_{i=1}^{N} \sum_{t=0}^{T-1} p_m(g_i(t)) \tag{15}$$

where $g_i(t)$ is the size of firm i at time t. Notice that this second method depends on the "branching" model to obtain an estimate of the probabilities $P(M_0 < M < M_1)$. A comparison between the observed P^{ob} and the predicted P^{th} distributions provides a way to tune the residual degree of freedom, and an indication of the goodness of the model. As showed in Fig. 8 the accordance, considering the extreme simplicity of the model, is surprisingly good.

5 Conclusions and research implication

We have shown that the relation between the variance of growth and the size of the firm, which constitutes the clearest and more often reported violation of the Gibrat Law, reduces to a diversification effect: bigger firms operate in more sub-markets and the variance of their growth is consequently reduced[11]. Moreover we have shown that in the industry under analysis, other possible effects, in particular the correlation among sub-markets and the dependence of some "strategic" diversification pattern of a firm on its size, if existing, are negligible.

The actual structure of the firm diversification can be described with good accuracy using the simple stochastic process proposed in Sec. 4, which embodies the idea that the scope of corporate diversification is shaped and constrained by "the things a firm already knows how to do". The model is well in tune with "Penrosian" competence-based interpretation of corporate diversification (Penrose, 1958) and with evolutionary ones whereby firms progressively learn how to innovate (cf. Dosi et al. (1995)) and also how to diversify, no matter what the object of this learning might be. In this work, due to industry at hand, we have emphasized *technological* capabilities and learning. However, the penetration of a firm in new submarkets might well be driven by learning in other domains. Our model would continue to apply in so far two general assumptions continue to hold, namely, first, the existence of some sort of corporate competencies providing the firm with the ability to diversify its business, and second, that these competencies (and so the ability to enter a new sub-market) increase with the number of times they have been effectively used (and so with the number of opened sub-markets)[12].

Note also that the observed relationship between the growth variance and the size of the firms, being milder than the LLN prediction, gives evidence against the interpretation of

[11]Incidentally, the observed relation between the number of sub-markets a firm posses and its size constitutes an original example of a "scaling" relation, and has to be added to other more famous example pertaining the domain of economics (for a review and critical discussion see Brock (1999))

[12]Note that in the model the cumulative effect on "competencies" is described by a linear function. In general one may refine the model using more sophisticated relations, for instance by dropping the strictly binary nature of the branching or introducing the possibility of a branch "death". This "finer" modeling would require, however, a higher "phenomenological" justification from the data.

"diversification" as a risk minimization strategy: indeed if this would be the case, firms would have to be present on much more sub-markets then they actually are. Rather, our evidence suggest that the scope of diversification is limited by the competencies a firm inherits from its past, which are accumulated at a pace less than proportional to firm size. Conversely, contrary to Penrose's suggestion, there appears to be no size-related "limits to diversification" (Penrose, 1958).

The foregoing analysis can be extended in several directions. First, the very similarity between our results on a single industry and those obtained on manufacturing as a whole (Stanley et al., 1997; Lee et al., 1998) is a puzzle in its own right: to what extent different industries (and technologies) differ in terms of their propensity to diversify?

Second, and relatedly, a natural way forward is to condition diversification patterns upon direct proxies of corporate competences, in that moving some steps away from pure stochastic descriptions of firms defined as rather amorphous, identical entities.

APPENDIX

A Solution of the Yule process

To simplify the solution of (9) let me introduce a rescaled size $t = \lambda(g - g_0)$ and consider, instead of the probabilities in (7), the variables $y_n(t)$ defined by

$$p_n(t) = e^{-nt}\, y_n(t) \; . \tag{16}$$

In these new variables the set of equation in (9) reduces to:

$$\begin{array}{rcll} y'_n(t) & = & (n-1)\, e^t\, y_{n-1}(t) & n > n_0 \\ y'_{n_0}(t) & = & 0 & n = n_0 \; . \end{array} \tag{17}$$

with initial conditions:

$$y_n(0) = \begin{cases} 1, & n = n_0 \\ 0, & n \neq n_0 \end{cases} \tag{18}$$

From (17), iterating over the index n, it is immediate to write the solution as multiple integral:

$$y_n(t) = (n-1)\,(n-2)\ldots n_0 \int_0^t dt_1 \int_0^{t1} dt_2 \ldots \int_0^{t_{n-n_0+1}} dt_{n-n_0} e^{t_1+t_2+\ldots+t_{n-n_0}} \tag{19}$$

Notice that, due to the complete symmetry of the integrand, the multiple integral over the $n - n_0$-dimensional hyper-cube of side t with the constraints $t_{n-n_0} < \ldots < t_2 < t_1 < t$ reduces to the integral over the whole hypercube divided by all the possible ordering of the constraints, which are $(n - n_0)!$. One thus obtains:

$$y_n(t) = \binom{n-1}{n-n_0} (e^t - 1)^{n-n_0} \; , \tag{20}$$

that, remembering the factorization in (16) and substituting t with its previous definition, reduces to (11) after obvious algebra.

References

Amaral, L.A.N., Buldyrev, S.V., Havlin, S., Salinger, M.A., Stanley, H.E. (1998) "Power Law Scaling for a System of Interacting Units with Complex Internal Structure", Physics Review Letters, 80, 1385-1388.

Boeri, T., (1989) "Does Firm Size Matter?", Giornale degli Economisti e Annali di Economia, 48, 477-495.

Bottazzi G., Dosi G., Lippi M., Pammolli F., Riccaboni M., (2000) "Processes of corporate growth in the evolution of an innovation-driven industry: the case of pharmaceutical" LEM Working Paper n. 2000/05, Sant'Anna Shool for Advanced Studies.

Bottazzi G., Dosi G., Lippi M., Pammolli F., Riccaboni M., (2001) "Innovation and Corporate Growth in the Evolution of the Drug Industry" International Journal of Industrial Organization, 19, 1161-1187.

Brock W., (1999) "Scaling in Economics: A Reader's Guide", Industrial and Corporate Change, 8, 409-446.

Dosi G. (1988) "Sources, procedures, and Microeconomics Effects of Innovation" Journal of Economic Literature, 26, 1120-1171.

Dosi G., Marsili O., Orsenigo L., Salvatore R., (1995) "Learning, Market Selection and the Evolution of Industrial Structures" Small Business Economics, 7, 411-436.

Evans, D., (1987) "Tests of Alternative Theories of Firm Growth', Journal of Industrial Economics, 95,657-674.

Feller W. (1968) "An Intrduction to Probabilty Theory and Its Applications" Wiley & Sons, New York.

Geroski P. A., (2000), "The Growth of Firms in Theory and Practice" in Foss N. and Malinke V. eds., New directions in economic strategy research, Oxford University Press, Oxford.

Gibrat R., (1931) "Les inégalités économiques", Librairie du Recueil Sirey, Paris.

Greiner L., (1972) "Evolution and Revolution as Organizations grow", Harvard Business Review, 50, 37-46.

Hall B. H., (1987) "The Relationship Between Firm Size and Firm Growth in the US Manufacturing Sector", Journal of Industrial Economics, 35, 583-606.

Hart P.E., Prais S. J., (1956) "The Analysis of Business Concentration: A Statistical Approach", Journal of the Royal Statistical Society, 119 (Ser. A), 150-191.

Hart P., (1962) "The Size and Growth of Firms", Economica, 29, 29-39.

Hymer S., Pashigian P. (1962) "Firm Size and Rate of Growth", Journal of Political Economy, 72, 556-569.

Ijiri y., Simon H.A., (1977) "Skew distributions and the sizes of business firms", North Holland, New York.

Lee Y., Nunes Amaral L. A., Canning D., Mayer M. and Stanley H. E. (1998) "Universal Features in the Growth Dynamics of Complex Organizations" Physical Review Letters, 81, 3275-3278.

Mansfield D. E., (1962) "Entry, Gibrat's Law, Innovation, and Growth of the Firms", American Economic Review, 52, 1024-1051.

Mueller D., (1972) "A Life Cycle Theory of the Firm", Journal of Industrial Economics, 20, 199-219.

Orsenigo L., Pammolli F., Riccaboni M., (2000) "Technological Change and Network Dynamics: Lessons from the Pharmaceutical Industry", Research Policy, 30, 485-508.

Penrose E., (1958) "The Theory of the Growth of the Firm", Oxford University Press, Oxford.

Simon H. and Bonini C., (1958) "The Size Distribution of business Firms", American Economic Review, 48, 607-617.

Sutton J., (1997) "Gibrat's Legacy" Journal of Economic Literature, 35, 40-59.

Sutton J., (1998) "Technology and Market Structure, Theory and History", MIT Press, Cambridge, MA.

Stanley M. H. R., Nunes Amaral L. A., Buldyrev S. V., Havlin S., Leschhorn, H., Maass P., Salinger M. A., Stanley H. E., (1997) "Scaling behavior in the growth of companies", Nature, 379, 804-806.

Snyder D.L. (1975) "Random Point Processes" Wiley & Sons, New York.

Teece, D., Rumelt, R., Dosi G. and Winter, S. (1994) "Understanding Corporate Coherence: Theory and Evidence" Journal of Economic Behavior and Organization, 23, 1-30.